

Original Article

Two-dimensional $N = (2, 2)$ dilaton supergravity from graded Poisson-Sigma models I

Complete actions and their symmetries

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Abstract. The formalism of graded Poisson-Sigma models allows the construction of $N = (2, 2)$ dilaton supergravity in terms of a minimal number of fields. For the gauged chiral $U(1)$ symmetry the full action, involving all fermionic contributions, is derived. The twisted chiral case follows by simple redefinition of fields. The equivalence of our approach to the standard second order one in terms of superfields is presented, although for the latter so far only the bosonic part of the action seems to have been available in the literature. It is shown how ungauged models can be obtained in a systematic way and some relations to relevant literature in superstring theory are discussed.

1 Introduction

Motivated mainly by (super-)string theory different versions of $N = (2, 2)$ supergravity in two dimensions [1, 2] have been studied extensively in superspace [3–9] some time ago. They also include extensions to dilaton theory [10]. Actual applications to string theory can be found more recently in [11–13]. A common drawback of these approaches is the extremely involved formulation, when the full machinery of (dilaton-) superfields is deployed. For this reason the results for component expansions obtained so far almost exclusively are restricted to the bosonic part of those theories.

On the other hand, already in bosonic $D = 2$ dilaton gravity the systematic use of Cartan variables in a “temporal” gauge [14–17] and the subsequent realization that essentially all general dilaton theories in two dimensions can be interpreted as a special case of a Poisson-Sigma model (PSM) [18–21] have led to a considerable number of new insights. Not only the extremely simple derivation of the classical solutions [17], but also a background independent quantization [16, 22, 23] has been possible. For a comprehensive review [24] can be recommended.

The extension to graded Poisson-Sigma models (gPSMs) has been equally successful [25–28]. As shown by the present authors a certain subclass of gPSMs, already identified as particularly attractive from the mathematical point of view in [29] by its dilaton-deformed super-Poincaré algebra, could be shown to be equivalent [30] to

the most relevant subclass of 2D dilaton $N = (1, 1)$ supergravity theories as formulated a long time ago by Park and Strominger [31]. This permitted the first complete solution (including fermionic fields) and the formulation of the superpoint particle in a gPSM background [30], as well as a complete classification of $N = (1, 1)$ solutions retaining certain supersymmetries [32] (BPS states). In the last reference also the problem of (non-minimal) coupling of conformal matter to those supergravities has been solved. Quantization following the same strategy as in the bosonic case is possible as well [33, 34]. The much richer structure of extended supergravities, encountered already in previous work on this subject [1–13], strongly motivates the application of gPSM technology to $N = (2, 2)$. As shown in our present paper this approach indeed is very successful and leads to novel insights.

In Sect. 2 we recall the main features of the gPSM formalism, together with the straightforward implementation of the field content for $N = (2, 2)$ supergravity. Among the two $U(1)$ symmetries the gauging of the chiral or of the twisted chiral case appear as simple alternatives.

Guided by the success of the special “dilaton prepotential supergravity” gPSM in the treatment for $N = (1, 1)$ we immediately concentrate on that in Sect. 3—not pursuing the involved elimination process employed for $N = (1, 1)$ [29]. Indeed already that subclass of $N = (2, 2)$ gPSMs is eventually found to be equivalent to the one proposed previously in the superfield approach [10].

Section 4 is devoted to the derivation of the Poisson tensor for generalized chiral $N = (2, 2)$ gPSM supergravity whereas in Sect. 5 we show how to reduce the twisted chiral case to the one of the previous section by simple redefinition of fields (mirror symmetry).

In Sect. 6 we first present the complete $N = (2, 2)$ supergravity action, involving all fermionic contributions for the chiral case, with the twisted chiral one to be obtained in an analogous fashion (Sect. 6.1). Finally the equivalence to the results obtained in the superfield formulation [10], where a second order action is used, is the subject of Sect. 6.2.

In Sect. 7 we study the formulation of ungauged supergravity in terms of gPSMs. Some models are derived explicitly, however interesting questions about the interpretation of the results cannot be answered conclusively as yet.

A summary of our results, their relation to previous ones in different approaches and an outlook concerning some obvious further applications are contained in Sect. 8. In Appendix Appendix A: we define our notations.

2 gPSM for $N = (2, 2)$ supergravity

A general gPSM consists of scalar fields $X^I(x)$, which themselves are coordinates of a graded Poisson manifold with Poisson tensor $P^{IJ}(X) = (-1)^{I+1}P^{JI}(X)$. The index I , in the generic case, includes commuting as well as anti-commuting fields¹. In addition one introduces the gauge potential $A = dX^I A_I = dX^I A_{mI}(x) dx^m$, a one form with respect to the Poisson structure as well as with respect to the 2d worldsheet coordinates. The gPSM action reads²

$$\begin{aligned} \mathcal{S}_{gPSM} &= \int_M dX^I \wedge A_I + \frac{1}{2} P^{IJ} A_J \wedge A_I \\ &= \int e(\partial_0 X^I A_{1I} - \partial_1 X^I A_{0I} + P^{IJ} A_{0J} A_{1I}) d^2x . \end{aligned} \quad (1)$$

The Poisson tensor P^{IJ} must have vanishing Nijenhuis tensor (obey a Jacobi-type identity with respect to the Schouten bracket related as $\{X^I, X^J\} = P^{IJ}$ to the Poisson tensor)

$$J^{IJK} = P^{IL} \partial_L P^{JK} + g\text{-perm}(IJK) = 0 , \quad (2)$$

where the sum runs over the graded permutations. The variation of A_I and X^I in (1) yields the gPSM field equations

$$dX^I + P^{IJ} A_J = 0 , \quad (3)$$

¹ The usage of different indices as well as other features of our notation are explained in Appendix Appendix A: For further details one should consult [28, 35].

² If the multiplication of forms is evident in what follows, the wedge symbol will be omitted.

$$dA_I + \frac{1}{2} (\partial_I P^{JK}) A_K A_J = 0 . \quad (4)$$

Due to (2) the action (1) is invariant under the symmetry transformations

$$\delta X^I = P^{IJ} \varepsilon_J , \quad \delta A_I = -d\varepsilon_I - (\partial_I P^{JK}) \varepsilon_K A_J , \quad (5)$$

where the term $d\varepsilon_I$ in the second of these equations provides the justification for calling A_I ‘‘gauge fields’’.

For a generic (g)PSM the commutator of two transformations (5) is a symmetry modulo the equations of motion (e.o.m.-s) in (3) only:

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] X^I = \delta_{\varepsilon_3} X^I \quad (6)$$

$$\begin{aligned} [\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] A_I &= \delta_{\varepsilon_3} A_I \\ &+ (dX^J + P^{JK} A_K) \partial_J \partial_I P^{RS} \varepsilon_{1S} \varepsilon_{2R} \end{aligned} \quad (7)$$

Here ε_3 is the new symmetry parameter

$$\varepsilon_{3I} = \partial_I P^{JK} \varepsilon_{1K} \varepsilon_{2J} + P^{JK} (\varepsilon_{1K} \partial_J \varepsilon_{2I} - \varepsilon_{2K} \partial_J \varepsilon_{1I}) \quad (8)$$

and from this equation it is seen that a generic gPSM obeys a non-linear algebra with structure functions $\partial_I P^{JK}$.

Only for P^{IJ} linear in X^I a closed Lie algebra is obtained, and (2) reduces to the Jacobi identity for the structure constants of a Lie group. If the Poisson tensor has a non-vanishing kernel there exist (one or more) Casimir functions $C(X)$ obeying

$$\{X^I, C\} = P^{IJ} \frac{\partial C}{\partial X^J} = 0 , \quad (9)$$

which, when the X^I obey the field equations of motion, are constants of motion.

In an abstract mathematical sense a gPSM is fully determined by the choice of its target space, i.e. the number of (bosonic and fermionic) target space variables and the number of (bosonic and fermionic) Casimirs. This statement is equivalent to the (local) existence of Casimir-Darboux coordinates. The situation in an application to (super-) gravity is less trivial: Here we need additional structure (a line-element or a point-particle) and global aspects with respect to that structure become relevant. Thus we cannot avoid solving the non-linear identity (2) for a particular Poisson tensor, which describes (super-)gravity in an explicit manner. A possible way to implement such a constraint in purely bosonic gravity consists in choosing the target-space variables

$$X^i = (X^\phi, X^a) = (\phi, X^a) \quad (10)$$

and the gauge fields

$$A_i = (A_\phi, A_a) = (\omega, e_a) \quad (11)$$

as the pairs dilaton/spin-connection and auxiliary-vector/zweibein (for a discussion of possible generalizations see [36]). Local Lorentz invariance then fixes the component

$$P^{a\phi} = X^b \varepsilon_b^a \quad (12)$$

of the Poisson tensor and the line element follows from the definition of a symmetric structure η^{ab} . This concept straightforwardly generalizes to $N = (1, 1)$ supergravity [28, 29]: one adds a pair of Majorana spinors (dilatino and gravitino) $X^\alpha = \chi^\alpha$ and $A_\alpha = \psi_\alpha$ to the target space variables and gauge fields resp. and one demands

$$P^{\alpha\phi} = -\frac{1}{2}\chi^\beta\gamma_{*\beta}^\alpha. \quad (13)$$

The limit of rigid supersymmetry in the flat space constrains the value of the purely fermionic part of the Poisson tensor:

$$P^{\alpha\beta} = -2iX^c\gamma_c^{\alpha\beta} + \text{terms} \propto \gamma_* \quad (14)$$

A more detailed implementation of this constraint, especially taking care of eventual singularities at $Y = X^a X_a/2 = 0$, has been discussed in [29].

Prepared with this knowledge about simpler models we can outline the principle steps that lead to a gPSM formulation of $N = (2, 2)$ supergravity:

Choice of the target space: First of all we have to determine the number of fields of our theory or, equivalently, the number of (local) gPSM symmetries. Certainly the bosonic variables must include ϕ and X^a from (10). As we are dealing with two supersymmetries, we need two pairs of Majorana dilatini ($\chi_\alpha^1, \chi_\alpha^2$) and gravitini ($\psi_\alpha^1, \psi_\alpha^2$) which we combine to complex Dirac spinors:

$$\chi_\alpha = \frac{1}{\sqrt{2}}(\chi_\alpha^1 - i\chi_\alpha^2) \quad \psi_\alpha = \frac{1}{\sqrt{2}}(\psi_\alpha^1 + i\psi_\alpha^2) \quad (15)$$

In addition, the $N = (2, 2)$ super-algebra has an internal $U(1)_V \times U(1)_A$ symmetry and none [11], one or both of these $U(1)$ factors can be gauged [1]. Of course, each gauged $U(1)$ leads to an additional scalar field, appearing as target-space variable of the gPSM. Most intuitive is the choice of one gauged $U(1)$: Beside the fields X^a and ω , which are eliminated to obtain a second order formulation (cf. 6.2), the target space variables then describe the field content of a $N = (2, 2)$ matter multiplet, while the gauge fields can be viewed as the components of the $N = (2, 2)$ multiplet comprising the zweibein e_a . Thus we concentrate on that case, which in addition has several advantages: In contrast to the case of two gauged $U(1)$ factors, this is an irreducible representation of supersymmetry³. On the other hand, the gauging of one factor reduces the number of invariant terms in the Poisson tensor which simplifies the highly non-trivial step of finding a solution to the condition (2). Finally, the dilaton supergravity formulated previously [10] deals with one gauged $U(1)$ as well and thus this choice will allow a comparison with that work.

Therefore, an additional pair of bosonic variables (π, B) is added to the target-space and the gauge-fields

resp., which leads to the final set of fields

$$X^I = (\phi, \pi, X^a, \chi^\alpha, \bar{\chi}^\alpha), \quad A_I = (\omega, B, e_a, \psi_\alpha, \bar{\psi}_\alpha). \quad (16)$$

Symmetry constraints: The invariance with respect to local Lorentz and B -gauge symmetry are fully determined by the underlying super-algebra. Local Lorentz invariance fixes the $P^{I\phi}$ components to

$$P^{a\phi} = X^b \epsilon_b^a, \quad P^{\pi\phi} = 0, \quad (17)$$

$$P^{\alpha\phi} = -\frac{1}{2}\chi^\beta\gamma_{*\beta}^\alpha, \quad P^{\bar{\alpha}\phi} = -\frac{1}{2}\bar{\chi}^\beta\gamma_{*\beta}^\alpha, \quad (18)$$

in order to create the appropriate covariant derivatives (cf. (22)-(24) below).

To determine the $P^{I\pi}$ components, a specific choice of gauging must be made. We first concentrate on chiral supergravity (gauged $U(1)_V$), which implies the choice

$$P^{a\pi} = 0, \quad P^{\alpha\pi} = -\frac{i}{2}\chi^\beta\gamma_{*\beta}^\alpha, \quad P^{\bar{\alpha}\pi} = \frac{i}{2}\bar{\chi}^\beta\gamma_{*\beta}^\alpha. \quad (19)$$

In Sect. 5 we will show how one obtains other gaugings from this specific result. Also the ungauged theory can be considered as a restricted version (Sect. 7). Finally we implement local supersymmetry in analogy to (14)

$$P^{\alpha\bar{\beta}} = -2iX^c\gamma_c^{\alpha\beta} + \text{terms} \propto \gamma_* \quad (20)$$

Rigid supersymmetry: With the result obtained so far the action of an $N = (2, 2)$ gPSM may be written as

$$\mathcal{S} = \int_M \left(\phi d\omega + \pi dB + X^a D e_a + \chi^\alpha D \psi_\alpha + \bar{\chi}^\alpha D \bar{\psi}_\alpha + iX^a (\psi \gamma_a \bar{\psi}) + \frac{1}{2} \hat{P}^{AB} A_B A_A \right), \quad (21)$$

where \hat{P}^{AB} is the $A = (a, \alpha, \bar{\alpha})$ part of the Poisson tensor without the specific contribution of (20). Setting $\hat{P}^{AB} = 0$ it is found that the remaining components with the covariant derivatives

$$D e_a = d e_a + \omega \epsilon_a^b e_b, \quad (22)$$

$$D \psi_\alpha = d \psi_\alpha - \frac{1}{2}(\omega + iB) \gamma_{*\alpha}^\beta \psi_\beta, \quad (23)$$

$$D \bar{\psi}_\alpha = d \bar{\psi}_\alpha - \frac{1}{2}(\omega - iB) \gamma_{*\alpha}^\beta \bar{\psi}_\beta \quad (24)$$

obey (2) and thus are a gPSM. As may be checked straightforwardly, the structure functions in (8) of this Poisson tensor exactly yield rigid $N = (2, 2)$ supersymmetry on flat space including the $U(1)_V$ factor of the internal symmetry group. This is an important consistency check of the setup discussed so far. The necessary step to be performed in the next section consists in finding a Poisson tensor with non-trivial bosonic potential P^{ab} and thus referring to a theory of supergravity.

³ Certainly this is not independent of the observation that the gPSM fields fit into $N = (2, 2)$ multiplets.

3 $N = (2, 2)$ dilaton prepotential supergravity

To find supergravity models with non-trivial bosonic potential we could perform similar steps as in [28]: Those components not yet fixed by (17)-(20) can be expanded in terms of Lorentz and B -gauge invariant functions and then the non-linear Jacobi identity (2) is solved order by order in the fermions χ_α . But, first, it is already known from the $N = (1, 1)$ case that the solution will not be unique for a given bosonic potential and, second, the expansion shows that the complexity of this task is most likely almost unmanageable. Fortunately, a different route suggests itself from the results of the $N = (1, 1)$ theories [29, 30]: The identity (2) is solved for a very special (simple) model only, more complicated theories are found by means of target space diffeomorphisms [28] that respect the constraints (17)-(20), especially conformal transformations.

As a simplified theory the $N = (2, 2)$ version of the model considered in [26] is chosen, cf. also Sects. (5.4) and (7.4) of [28] as well as [29, 30]. The idea is to minimize in a first step the contributions to torsion, in other words the dependence on X^a . Thus it is demanded that the Poisson tensor is independent of X^a except for the minimal contributions in (17) and (20). Then the expansion of the bosonic potential $P^{ab} = \epsilon^{ab}V$ reduces to

$$V = v + \frac{1}{2}\chi^2 v_2 + \frac{1}{2}\bar{\chi}^2 \bar{v}_2 + \frac{1}{4}\chi^2 \bar{\chi}^2 v_4, \quad (25)$$

where the remaining functions depend on ϕ and π only. For convenience we introduce the notation $X = \phi + i\pi$ and thus $v = v(X, \bar{X})$ etc.⁴ In the mixed component of the Poisson tensor

$$P^{a\alpha} = F_S^a \chi^\alpha + F_P^a (\chi \gamma_*)^\alpha + F^{ab} (\bar{\chi} \gamma_b)^\alpha \quad (26)$$

F_S^a and F_P^a include contributions proportional to $\chi \gamma^a \bar{\chi}$ and $\chi \gamma^a \gamma_* \bar{\chi}$ only, which by means of Fierz identities can be transformed into contributions that already appear in F^{ab} . Therefore, (26) reduces to

$$P^{a\alpha} = F^{ab} (\bar{\chi} \gamma_b)^\alpha = (F_{(s)} \eta^{ab} + F_{(a)} \epsilon^{ab}) (\bar{\chi} \gamma_b)^\alpha, \quad (27)$$

$$F_{(s)} = f_{(s)} + \frac{1}{2}\chi^2 \tilde{f}_{(s)}, \quad F_{(a)} = f_{(a)} + \frac{1}{2}\chi^2 \tilde{f}_{(a)}. \quad (28)$$

According to our conventions $P^{a\bar{\alpha}} = -(P^{a\alpha})^*$. Finally, the purely fermionic terms must be considered. $P^{\alpha\bar{\beta}}$ for chiral gaugings of $U(1)$ has no invariant term proportional to γ_* , while $P^{\alpha\beta} = U \gamma_*^{\alpha\beta}$ with

$$U = u + \frac{1}{2}\chi^2 u_2 + \frac{1}{2}\bar{\chi}^2 \tilde{u}_2 + \frac{1}{4}\chi^2 \bar{\chi}^2 u_4. \quad (29)$$

Notice that \tilde{u}_2 need not be the complex conjugate of u_2 .

⁴ Of course, the action of chiral supergravity is most conveniently written in terms of a complex scalar field X and a complex gauge field $\omega + iB$. However, we keep ω and B separate to simplify the generalization to different gaugings.

The implementation of the condition $J^{JK} = 0$ in (2) is a straightforward, but still tedious calculation. All identities with at least one ϕ or π are taken into account by the invariant expansions; identities with an odd number of bosonic indices contribute to even (zero, two, four) degrees in the number of fermions, the other ones to odd (one, three) degrees. We introduce the notation $f(\phi, \pi)' = \partial f(\phi, \pi)/\partial \phi$ and $\dot{f}(\phi, \pi) = \partial f(\phi, \pi)/\partial \pi$. The results of that calculation can be summarized as follows:

order zero: $J^{abc} = 0$ is the purely bosonic identity and automatically satisfied.

$J^{a\alpha\beta} = 0$ relates $P^{a\alpha}$ to $P^{\alpha\beta}$:

$$f_{(s)} = \frac{i}{4} u', \quad f_{(a)} = 0. \quad (30)$$

$J^{a\alpha\bar{\beta}} = 0$ expresses the bosonic potential v in terms of u , which by

$$v = -\frac{1}{8}(\bar{u}u)' \quad (31)$$

defines it to be a prepotential.

order one: The $\bar{\chi}$ contribution from $J^{ab\alpha}$ yields

$$\bar{v}_2 = \frac{1}{8} u'' . \quad (32)$$

The χ contribution of $J^{ab\alpha}$ as well as $J^{\alpha\beta\bar{\gamma}}$ constrain the dependence of the prepotential on ϕ and π :

$$u = u(\phi + i\pi) = u(X) \quad (33)$$

Finally, we obtain from $J^{\alpha\beta\gamma} = 0$ that $u_2 = \tilde{u}_2 = 0$.

order two: $J^{abc} = 0$ is again trivial, while $J^{a\alpha\beta}$ and $J^{a\alpha\bar{\beta}}$ set the higher order contributions of $P^{a\alpha}$ to zero: $\tilde{f}_{(s)} = \tilde{f}_{(a)} = 0$.

orders three and four: The remaining identities are now almost trivial. J^{abc} has one non-vanishing term of order three, which tells us that $v_4 = 0$. Similarly one gets from $J^{\alpha\beta\bar{\gamma}}$ that $u_4 = 0$. All remaining identities are then automatically satisfied.

Putting the pieces together, the Poisson tensor apart from the components in (17)-(19) becomes

$$P^{ab} = \epsilon^{ab} \left(-\frac{1}{8}(\bar{u}u)' + \frac{1}{16}\bar{u}''\chi^2 + \frac{1}{16}u''\bar{\chi}^2 \right), \quad (34)$$

$$P^{a\alpha} = \frac{i}{4} u' (\bar{\chi} \gamma^a)^\alpha, \quad P^{a\bar{\alpha}} = \frac{i}{4} \bar{u}' (\chi \gamma^a)^\alpha, \quad (35)$$

$$P^{\alpha\bar{\beta}} = -2i X^a (\gamma_a)^{\alpha\bar{\beta}}, \quad (36)$$

$$P^{\alpha\beta} = u \gamma_*^{\alpha\beta}, \quad P^{\bar{\alpha}\bar{\beta}} = \bar{u} \gamma_*^{\bar{\alpha}\bar{\beta}}. \quad (37)$$

The similarity of this tensor to the related model with $N = (1, 1)$ supersymmetry [cf. (5.34)–(5.36) in [28]] is obvious.

As mentioned already in Sect. 2 the knowledge of eventual Casimir functions (9) is very important. In case of bosonic gravity or $N = (1, 1)$ supergravity, the bosonic part of the Poisson tensor has odd dimension and thus there exists at least one Casimir function. Here the bosonic part has even dimension, but the symmetry constraints

imply that it can never have full rank. Thus there exist at least two Casimirs. One of them can be chosen as the $N = (2, 2)$ extension of the one present in any PSM gravity model:

$$C = 8Y - \bar{u}u + \frac{1}{2}\chi^2\bar{u}' + \frac{1}{2}\bar{\chi}^2u' \quad (38)$$

The second one is related to the new gauge symmetry. As all bosonic fields are singlets under the B -gauge transformation its body is simply π . Indeed, a straightforward calculation shows that

$$C_\pi = \pi + \frac{i\bar{u}}{4C}\chi^2 - \frac{i u}{4C}\bar{\chi}^2 - \frac{1}{C}X^a(\chi\gamma_a\gamma_*\bar{\chi}) \quad (39)$$

commutes in the sense of (9) with all target space variables. For ground-state configurations ($C = 0$ in (38), cf. [32]) with non-vanishing fermion fields the form (39) of the second Casimir function needs not be well defined. This problem finds a resolution within the study of the integrability of the theory [37]. In certain cases additional (fermionic or bosonic) Casimir functions can appear (for $N = (1, 1)$ cf. [28]).

4 General chiral supergravity

For many applications the model (34)-(37) is not yet general enough, as its bosonic potential v has been restricted to be independent of Y . In the case of $N = (1, 1)$ supergravity the present authors have found [29, 30] that all “genuine” supergravities—i.e. gPSM theories which obey symmetry restrictions as pointed out in Sect. 2—are obtained from a model of the type (34)-(37) by the use of a field-dependent conformal transformation. This concept requires a generalization in the present case, as the theory depends on a complex scalar X instead of the real dilaton ϕ . Thus, we are looking for a target space diffeomorphism (cf. [28])

$$X^I \implies \hat{X}^I = \hat{X}^I(X, \bar{X}) = \hat{X}^I(\phi, \pi), \quad (40)$$

which yields new gauge potentials and a new Poisson tensor

$$\begin{aligned} \hat{A}_I &= \frac{\partial X^J}{\partial \hat{X}^I} A_J, \\ \hat{P}^{IJ} &= (-1)^{K(I+1)} (\partial_K \hat{X}^I) P^{KL} (\partial_L \hat{X}^J), \end{aligned} \quad (41)$$

but leaves the symmetry constraints (17)-(20) invariant. We do not intend to solve this constraint in full generality, but use the result from [29], namely that these target space diffeomorphisms can be interpreted as conformal transformations.

Consider a generalized conformal transformation that depends on a generic complex function of $Q(\varphi, \pi)$. From the constraints that X^a is a real field the transformation must be of the form

$$\hat{X} = X, \quad \hat{X}^a = e^{-(Q+\bar{Q})/4} X^a, \quad (42)$$

$$\hat{\chi}^\alpha = e^{-\bar{Q}/4} \chi^\alpha, \quad \hat{\bar{\chi}}^\alpha = e^{-Q/4} \bar{\chi}^\alpha. \quad (43)$$

This transformation leaves the components (17)-(19) invariant, while $\hat{P}^{\alpha\bar{\beta}}$ in terms of the fields without hats becomes

$$\begin{aligned} \hat{P}^{\alpha\bar{\beta}} &= e^{-(Q+\bar{Q})/4} P^{\alpha\bar{\beta}} \\ &+ \frac{1}{16} e^{-(Q+\bar{Q})/4} \left((\bar{\chi}\chi\gamma_*^{\alpha\beta} - \bar{\chi}\gamma_*\chi\epsilon^{\alpha\beta})(Q' + \bar{Q}' + i\dot{Q} - i\dot{\bar{Q}}) \right. \\ &\quad \left. - \bar{\chi}\gamma^a\chi(\gamma_*\gamma_a)^{\alpha\beta}(Q' - \bar{Q}' + i\dot{Q} + i\dot{\bar{Q}}) \right). \end{aligned} \quad (44)$$

Clearly the last line has to vanish if local supersymmetry shall still be implemented by (20). The possible solutions are $Q(\phi + i\pi)$ analytic, $Q(\phi)$ real or $Q(\pi)$ imaginary. The first possibility has an analogue in superspace formulation of chiral $N = (2, 2)$ supergravity [1]: infinitesimal super-Weyl transformations preserving the constraints may be written as

$$E_a^M \delta E_M^b = \delta_a^b (\Lambda + \bar{\Lambda}), \quad E_\alpha^M \delta E_M^\beta = \delta_\alpha^\beta \bar{\Lambda}, \quad (45)$$

$$E_a^M \delta E_M^{\bar{\beta}} = i\gamma_a^{\beta\gamma} D_\gamma \Lambda, \quad (46)$$

with a *chiral* transformation parameter Λ . Therefore, in the superspace formulation the gravitino—the lowest component of E_m^α —transforms with a function that depends on the anti-chiral field $\bar{\Phi} = \phi - i\pi + \dots$. The remaining two possibilities have no obvious analogue in superspace, but real $Q(\phi)$ nonetheless is reminiscent of the case of non-minimally gauged supergravity [(28) in [1]].

To follow as close as possible the philosophy of the superspace formulation $Q(X)$ is chosen as an analytic function in X , the remaining possibilities are certainly interesting but we leave their investigation for future work. This choice in turn implies by (41) with $Z = Q'$

$$\hat{\omega} = \omega + \frac{1}{4} ((Z + \bar{Z})X^b e_b + \bar{Z}\chi\psi + Z\bar{\chi}\bar{\psi}), \quad (47)$$

$$\hat{B} = B - \frac{i}{4} (\bar{Z}\chi\psi - Z\bar{\chi}\bar{\psi}), \quad (48)$$

$$\hat{e}_a = e^{(Q+\bar{Q})/4} e_a, \quad \hat{\psi}_\alpha = e^{\bar{Q}/4} \psi_\alpha, \quad \hat{\bar{\psi}}_\alpha = e^{Q/4} \bar{\psi}_\alpha. \quad (49)$$

It is now straightforward to derive the new Poisson tensor in terms of the variables \hat{X}^I . By doing this it is found that the prepotential $u(X)$ transforms as

$$\hat{u} = e^{-\bar{Q}/2} u, \quad \dot{\hat{u}} = i(\dot{u}' + \bar{Z}\dot{\hat{u}}) \quad (50)$$

and thus \hat{u} no longer represents an analytic function.

To economize writing we drop the hats for the fields of the generalized Poisson tensor in the following and in addition introduce the new functions

$$w(X) = \frac{1}{4} e^{\bar{Q}/2} u, \quad W(X, \bar{X}) = -2w\bar{w}. \quad (51)$$

With (41) the generalized Poisson tensor becomes

$$\begin{aligned} P^{ab} &= \epsilon^{ab} \left(e^{-(Q+\bar{Q})/2} W' + \frac{1}{2} Y(Z + \bar{Z}) \right. \\ &\quad \left. + \frac{1}{4} \chi^2 e^{-Q/2} \bar{w}'' + \frac{1}{4} \bar{\chi}^2 e^{-\bar{Q}/2} w'' \right), \end{aligned} \quad (52)$$

$$P^{a\alpha} = ie^{-\bar{Q}/2} w' (\bar{\chi} \gamma^a)^\alpha - \frac{\bar{Z}}{4} X^b (\chi \gamma_b \gamma^a \gamma_*)^\alpha, \quad (53)$$

$$P^{a\bar{\alpha}} = ie^{-Q/2} \bar{w}' (\chi \gamma^a)^\alpha - \frac{Z}{4} X^b (\bar{\chi} \gamma_b \gamma^a \gamma_*)^\alpha, \quad (54)$$

$$P^{\alpha\bar{\beta}} = -2i X^a (\gamma_a)^{\alpha\beta}, \quad (55)$$

$$P^{\alpha\beta} = \left(u + \frac{\bar{Z}}{4} \chi^2 \right) \gamma_*^{\alpha\beta}, \quad P^{\bar{\alpha}\bar{\beta}} = \left(\bar{u} + \frac{Z}{4} \bar{\chi}^2 \right) \gamma_*^{\alpha\beta}, \quad (56)$$

yielding the Casimir functions (38) and (39) from (42) and (43)

$$C = 8(W + e^{(Q+\bar{Q})/2} (Y + \frac{1}{4} \chi^2 e^{-Q/2} \bar{w}' + \frac{1}{4} \bar{\chi}^2 e^{-\bar{Q}/2} w')) , \quad (57)$$

$$C_\pi = \pi + ie^{\bar{Q}/2} \frac{\bar{w}}{C} \chi^2 - ie^{Q/2} \frac{w}{C} \bar{\chi}^2 - \frac{e^{(Q+\bar{Q})/2}}{C} X^a (\chi \gamma_a \gamma_* \bar{\chi}) . \quad (58)$$

5 Twisted-chiral supergravity

To find other types of gaugings of the internal $U(1)_V \times U(1)_A$ symmetries the restrictions on allowed target-space diffeomorphisms is partially relaxed: We still insist that (17), (18), (20) and the first equation of (19) remain unchanged, but we allow changes in the second and third equation of (19). Nevertheless, we have to assume that the $P^{\alpha\pi}$ and $P^{\bar{\alpha}\pi}$ components define sensible covariant derivatives, i.e. they remain functions of χ^α and $\bar{\chi}^\alpha$ alone. From the covariant derivatives of ψ_α in chiral spinor components

$$(D\psi)_+ = \left(d - \frac{1}{2}(\omega + iB) \right) \psi_+, \quad (59)$$

$$(D\psi)_- = \left(d + \frac{1}{2}(\omega + iB) \right) \psi_- \quad (60)$$

and their hermitian conjugates the remaining transformations are obtained by the simple exchange

$$\chi^+ \longleftrightarrow \bar{\chi}^+, \quad \psi_+ \longleftrightarrow \bar{\psi}_+ \quad (61)$$

and

$$\chi^- \longleftrightarrow \bar{\chi}^-, \quad \psi_- \longleftrightarrow \bar{\psi}_-. \quad (62)$$

The combination of the two is obviously trivial, applying one of them yields the twisted chiral gauging. Notice that the supersymmetry transformation $X^a(\psi \wedge \gamma_a \psi)$ is invariant under (61) and/or (62).

In principle, all formulae of the previous Sections can be taken over together with the replacements $\chi^- \leftrightarrow \bar{\chi}^-$ and $\psi_- \leftrightarrow \bar{\psi}_-$ to describe the twisted-chiral version of supergravity. However, some expressions become rather

lengthy. The twisted-chiral analogue of the Poisson tensor (19) and (52)-(56) turns out to be

$$P^{a\pi} = 0, \quad P^{\alpha\pi} = -\frac{i}{2} \chi^\alpha, \quad P^{\bar{\alpha}\pi} = \frac{i}{2} \bar{\chi}^\alpha, \quad (63)$$

$$P^{ab} = \epsilon^{ab} \left(e^{-(Q+\bar{Q})/2} W' + \frac{1}{2} Y(Z + \bar{Z}) + \frac{1}{4} \chi \bar{\chi} (e^{-Q/2} \bar{w}'' + e^{-\bar{Q}/2} w'') + \frac{1}{4} \chi \gamma_* \bar{\chi} (e^{-Q/2} \bar{w}'' - e^{-\bar{Q}/2} w'') \right), \quad (64)$$

$$P^{a\alpha} = \frac{i}{2} (e^{-\bar{Q}/2} w' + e^{-Q/2} \bar{w}') (\chi \gamma^a)^\alpha + \frac{i}{2} (e^{-\bar{Q}/2} w' - e^{-Q/2} \bar{w}') (\chi \gamma^a \gamma_*)^\alpha - \frac{1}{8} ((\bar{Z} + Z) X^b \epsilon_b^a + (\bar{Z} - Z) X^a) \chi^\alpha - \frac{1}{8} ((\bar{Z} + Z) X^a + (\bar{Z} - Z) X^b \epsilon_b^a) (\chi \gamma_*)^\alpha, \quad (65)$$

$$P^{\alpha\bar{\beta}} = -2i X^a \gamma_a^{\alpha\beta} + \frac{1}{2} \left(u + \frac{\bar{Z}}{4} (\chi \bar{\chi} + \chi \gamma_* \bar{\chi}) \right) (\gamma_* - \epsilon)^{\alpha\beta} + \frac{1}{2} \left(\bar{u} + \frac{Z}{4} (\chi \bar{\chi} - \chi \gamma_* \bar{\chi}) \right) (\gamma_* + \epsilon)^{\alpha\beta}, \quad (66)$$

$$P^{\alpha\beta} = P^{\bar{\alpha}\bar{\beta}} = 0. \quad (67)$$

In analogy to (53)/(54) we have also $P^{a\bar{\alpha}} = -(P^{a\alpha})^*$. The symbol ϵ in (66) is the symplectic tensor used to raise spinor indices, $(\gamma_* \pm \epsilon)/2$ are the (anti-)chiral projection operators. Finally, the Casimir functions are obtained as

$$C = 8 \left(W + e^{(Q+\bar{Q})/2} \left(Y + \frac{1}{4} \chi \bar{\chi} (e^{-Q/2} \bar{w}' + e^{-\bar{Q}/2} w') + \frac{1}{4} \chi \gamma_* \bar{\chi} (e^{-Q/2} \bar{w}' - e^{-\bar{Q}/2} w') \right) \right), \quad (68)$$

$$C_\pi = \pi + i \frac{\chi \bar{\chi}}{C} (e^{\bar{Q}/2} \bar{w} - e^{Q/2} w) + i \frac{\chi \gamma_* \bar{\chi}}{C} (e^{\bar{Q}/2} \bar{w} + e^{Q/2} w) - \frac{e^{(Q+\bar{Q})/2}}{C} X^a (\chi \gamma_a \bar{\chi}). \quad (69)$$

It is important to notice that the discrete transformations (61) always are defined *globally*, in contrast to the conformal transformations considered in Sect. 4. Thus, the chiral and twisted-chiral models are physically equivalent. This is the well-known behavior of the geometrical (topological) sector under mirror symmetry, while propagating matter degrees of freedom in general are not invariant.

6 Actions for gauged dilaton supergravity

Having found the explicit Poisson tensors that describe (twisted-)chiral supergravity, we are now ready to write

down the corresponding supergravity actions and their symmetry transformations. Then these results are compared to the models known in literature, which are formulated in an (equivalent) second order derivative formulation.

6.1 gPSM action and its symmetries

From (1) together with (52)-(56) the full chiral dilaton supergravity action becomes

$$\begin{aligned} \mathcal{S}_{ch} = & \int_M \left(\phi d\omega + \pi dB + X^a D e_a + \chi^\alpha D \psi_\alpha + \bar{\chi}^\alpha D \bar{\psi}_\alpha \right. \\ & + \epsilon \left(\frac{1}{2} Y (Z + \bar{Z}) + e^{-(Q+\bar{Q})/2} W' \right. \\ & + \frac{1}{4} \chi^2 e^{-Q/2} \bar{w}'' + \frac{1}{4} \bar{\chi}^2 e^{-\bar{Q}/2} w'' \left. \right) \\ & + \frac{\bar{Z}}{4} X^a (\chi \gamma_a \gamma^b e_b \gamma_* \psi) + \frac{Z}{4} X^a (\bar{\chi} \gamma_a \gamma^b e_b \gamma_* \bar{\psi}) \\ & - i e^{-\bar{Q}/2} w' (\bar{\chi} \gamma^a e_a \psi) - i e^{-Q/2} \bar{w}' (\chi \gamma^a e_a \bar{\psi}) \\ & + 2i X^a \bar{\psi} \gamma_a \psi - \frac{1}{2} \left(u + \frac{\bar{Z}}{4} \chi^2 \right) \psi \gamma_* \psi - \frac{1}{2} \left(\bar{u} + \frac{Z}{4} \bar{\chi}^2 \right) \bar{\psi} \gamma_* \bar{\psi} \left. \right). \end{aligned} \quad (70)$$

The first two terms proportional ϵ , the two-dimensional volume form, contain the bosonic potential. This action is invariant under local Lorentz symmetry and B -gauge symmetry, which both are realized linearly according to (17)–(19) together with (5). A special field dependent choice of ε_a represents 2D diffeomorphisms, which we need not reproduce explicitly because of the manifest deffeomorphism invariance of (70). More important in different applications are the supersymmetry transformations. For the target-space variables the result

$$\delta X = \bar{\chi} \gamma_* \bar{\varepsilon}, \quad \delta \bar{X} = \chi \gamma_* \varepsilon, \quad (71)$$

$$\begin{aligned} \delta X^a = & -\frac{1}{4} X^b \left(\bar{Z} (\chi \gamma_b \gamma^a \gamma_* \varepsilon) + Z (\bar{\chi} \gamma_b \gamma^a \gamma_* \bar{\varepsilon}) \right) \\ & + i \left(e^{-\bar{Q}/2} w' (\bar{\chi} \gamma^a \varepsilon) + e^{-Q/2} \bar{w}' (\chi \gamma^a \bar{\varepsilon}) \right), \end{aligned} \quad (72)$$

$$\delta \chi^\alpha = 2i X^a (\bar{\varepsilon} \gamma_a)^\alpha - \left(u + \frac{\bar{Z}}{4} \chi^2 \right) (\varepsilon \gamma^a)^\alpha, \quad (73)$$

$$\delta \bar{\chi}^\alpha = 2i X^a (\varepsilon \gamma_a)^\alpha - \left(\bar{u} + \frac{Z}{4} \bar{\chi}^2 \right) (\bar{\varepsilon} \gamma^a)^\alpha, \quad (74)$$

is obtained, while the expressions for the gauge fields are more complicated:

$$\begin{aligned} \delta \omega = & \frac{1}{4} X^b \left(\bar{Z}' (\chi \gamma_b \gamma^a \gamma_* \varepsilon) + Z' (\bar{\chi} \gamma_b \gamma^a \gamma_* \bar{\varepsilon}) \right) \\ & - i \left((e^{-\bar{Q}/2} w')' (\bar{\chi} \gamma^a \varepsilon) + (e^{-Q/2} \bar{w}')' (\chi \gamma^a \bar{\varepsilon}) \right) \\ & - \left(u' + \frac{\bar{Z}'}{4} \chi^2 \right) \psi \gamma_* \varepsilon - \left(\bar{u}' + \frac{Z'}{4} \bar{\chi}^2 \right) \bar{\psi} \gamma_* \bar{\varepsilon} \end{aligned} \quad (75)$$

$$\begin{aligned} \delta B = & \frac{i}{4} X^b \left(Z' (\bar{\chi} \gamma_b \gamma^a \gamma_* \bar{\varepsilon}) - \bar{Z}' (\chi \gamma_b \gamma^a \gamma_* \varepsilon) \right) \\ & - i \left((e^{-\bar{Q}/2} w') \cdot (\bar{\chi} \gamma^a \varepsilon) + (e^{-Q/2} \bar{w}') \cdot (\chi \gamma^a \bar{\varepsilon}) \right) \\ & - i \left(u' + \bar{Z} u - \frac{\bar{Z}'}{4} \chi^2 \right) \psi \gamma_* \varepsilon \end{aligned} \quad (76)$$

$$\begin{aligned} & - i \left(\bar{u}' + Z \bar{u} - \frac{Z'}{4} \bar{\chi}^2 \right) \bar{\psi} \gamma_* \bar{\varepsilon} \\ \delta e_a = & \frac{1}{4} \left(\bar{Z} (\chi \gamma_a \gamma^b \gamma_* \varepsilon) + Z (\bar{\chi} \gamma_a \gamma^b \gamma_* \bar{\varepsilon}) \right) e_b \\ & + 2i (\psi \gamma_a \bar{\varepsilon} + \bar{\psi} \gamma_a \varepsilon) \end{aligned} \quad (77)$$

$$\begin{aligned} \delta \psi_\alpha = & -(D\varepsilon)_\alpha + \frac{\bar{Z}}{4} X^a (\gamma_a \gamma^b \gamma_* \varepsilon)_\alpha e_b \\ & + i e^{-Q/2} \bar{w}' (\gamma^a \bar{\varepsilon})_\alpha e_a - \frac{\bar{Z}}{4} \chi_\alpha (\psi \gamma_* \varepsilon) \end{aligned} \quad (78)$$

$$\begin{aligned} \delta \bar{\psi}_\alpha = & -(D\bar{\varepsilon})_\alpha + \frac{Z}{4} X^a (\gamma_a \gamma^b \gamma_* \bar{\varepsilon})_\alpha e_b \\ & + i e^{-\bar{Q}/2} w' (\gamma^a \varepsilon)_\alpha e_a - \frac{Z}{4} \bar{\chi}_\alpha (\bar{\psi} \gamma_* \bar{\varepsilon}) \end{aligned} \quad (79)$$

In the last two equations under B -gauge transformation the symmetry parameter ε_α behaves as ψ_α (cf. (23)), $\bar{\varepsilon}_\alpha$ as $\bar{\psi}_\alpha$. Again the similarity of the action (70) together with its supersymmetry transformations (75)–(79) and the result obtained in $N = (1, 1)$ supergravity [cf. [30] (18) and (20)–(25)] is immediate.

Twisted-chiral supergravity follows from the field reflections (62). As the corresponding formulae are quite lengthy, but can be reconstructed easily, we do not reproduce them here.

6.2 Relation to second-order formulation

For $N = (1, 1)$ dilaton supergravity a very detailed study of the relation between the gPSM-based (first-order) formulation and the second order formulation from superspace has been carried out by the present authors in [30]. Here we just want to sketch the basic steps for $N = (2, 2)$ supergravity that basically lead to an equivalent result. We restrict the explicit calculations to the case of chiral supergravity, the twisted chiral version again follows by a simple change of variables.

To make contact with the second order formulation of supergravity as it follows by integrating out auxiliary fields of superspace, also our auxiliary field X^a and the part of the spin-connection depending on bosonic torsion must be eliminated. The necessary steps have been worked out for $N = (1, 1)$ supergravities in detail in [28], Sect. 6.3. As the procedure is independent of the number of target-space variables as well as of the details of the Poisson tensor, all formulae can be carried over immediately to the present application.

Variation of the action (70) with respect to X^a yields the torsion equation and is used to eliminate the indepen-

dent spin connection according to

$$\omega_a = e_a^m \omega_m = \tilde{\omega}_a - \tilde{\tau}_a, \quad (80)$$

$$\tilde{\omega}_a = \epsilon^{mn} \partial_n e_{ma} - 2i \epsilon^{mn} (\bar{\psi}_n \gamma_a \psi_m), \quad (81)$$

$$\tilde{\tau}_a = -\frac{1}{2} (\partial_a \hat{P}^{AB}) \epsilon^{mn} e_{Bn} e_{Am}. \quad (82)$$

In the last equation \hat{P}^{AB} are the components (52)-(54) and (56) of the Poisson tensor without the minimal torsion contribution (55), which has been included in the definition of the supersymmetry covariant spin connection $\tilde{\omega}$. After replacing the independent spin-connection by (80)-(82) and a subsequent partial integration, the action can be varied with respect to X^a again. This finally allows to eliminate that field by a purely algebraic (and even only linear) equation:

$$X^a = -\epsilon^{an} \left(\partial_n \phi + \frac{1}{2} (\chi \gamma_* \psi_n) + \frac{1}{2} (\bar{\chi} \gamma_* \bar{\psi}_n) \right) \quad (83)$$

We introduce the curvature scalar and its partners as

$$\tilde{R} = 2 * d\tilde{\omega}, \quad \tilde{B} = * dB, \quad (84)$$

$$\tilde{\sigma}_\alpha = * (\tilde{D}\psi)_\alpha, \quad \tilde{\bar{\sigma}}_\alpha = * (\tilde{D}\bar{\psi})_\alpha, \quad (85)$$

where the covariant derivatives \tilde{D} are defined as in (23) and (24) with ω replaced by $\tilde{\omega}$. After some algebra the second-order version of the action (70) is found:

$$\begin{aligned} \mathcal{S}_{ch} = & \int d^2x e \left(\frac{1}{2} \tilde{R} \phi + \tilde{B} \pi + \chi^\alpha \tilde{\sigma}_\alpha + \bar{\chi}^\alpha \tilde{\bar{\sigma}}_\alpha \right. \\ & - \frac{1}{2} (Z + \bar{Z}) \partial^m \phi \partial_m \phi + e^{-(Q+\bar{Q})/2} W' \\ & + \frac{1}{4} (\chi^2 e^{-Q/2} \bar{w}'' + \bar{\chi}^2 e^{-\bar{Q}/2} w'') \\ & + i \epsilon^{mn} (e^{-\bar{Q}/2} w' (\bar{\chi} \gamma_n \psi_m) + e^{-Q/2} \bar{w}' (\chi \gamma_n \bar{\psi}_m)) \\ & - \left(\frac{Z}{8} (\partial^m \phi (\chi \gamma_* \psi_m) + 2 \epsilon^{mn} (\partial_n \phi) \bar{\chi} \bar{\psi}_m \right. \\ & \quad \left. - \epsilon^{mn} (\bar{\chi} \bar{\psi}_n) (\chi \gamma_* \psi_m)) + \text{h. c.} \right) \\ & + \frac{1}{32} (Z - \bar{Z}) (\chi^2 \psi^m \psi_m - \bar{\chi}^2 \bar{\psi}^m \bar{\psi}_m) \\ & \left. + \frac{1}{2} \epsilon^{mn} (u (\psi_n \gamma_* \psi_m) + \bar{u} (\bar{\psi}_n \gamma_* \bar{\psi}_m)) \right) \quad (86) \end{aligned}$$

From the results of [30] it is expected that also here this action is —up to some field redefinitions— equivalent to the model of [10], although in that work only the first two lines of (86) has been worked out explicitly. This is confirmed by several observations:

1. The bosonic potential of (86) is equivalent to the one of [10], and both are equivalent to the bosonic potentials of the $N = (1, 1)$ case.

2. There exists a kinetic term for ϕ but not for π . As pointed out in [30] this is a consequence of the first order formalism in terms of a gPSM. For the same reason, the gPSM based dilaton supergravity does not produce a kinetic term for the dilatino. However, by the field redefinition

$$\frac{\psi}{-m}^\alpha = \psi_m^\alpha - \frac{i}{8} \bar{Z} e_m^a \epsilon_{ab} (\bar{\chi} \gamma^b)^\alpha, \quad (87)$$

such a term is generated. This redefinition is necessary, as the conformal transformation (42)-(49) is not equivalent to a super-Weyl transformation in superspace [1]. There beside the multiplication with the conformal factor an additional term is needed to preserve the torsion constraints (cf. (45) and (46)). This generates the kinetic term for the fermions but does not affect the scalar fields.

If $Z = 0$ the kinetic term of the dilaton disappears. Such models are related to $N = (2, 2)$ supergravity described in terms of a single supergravity multiplet (e^a, ψ^α, B) [1, 2]. Indeed, on a patch with $u'' \neq 0$ the dilatino χ can be eliminated in that case as well. The resulting action

$$\begin{aligned} \mathcal{S}_{ch} = & \int d^2x e \left(\frac{1}{2} \tilde{R} \phi + \tilde{B} \pi - \frac{4}{u''} \tilde{\sigma}^2 - \frac{4}{u''} \tilde{\bar{\sigma}}^2 - \frac{1}{8} (\bar{u} u)' \right. \\ & + \frac{1}{2} (u \epsilon^{mn} \psi_n \gamma_* \psi_m + \bar{u} \epsilon^{mn} \bar{\psi}_n \gamma_* \bar{\psi}_m) \\ & - 2i \left(\frac{\bar{u}'}{u''} \epsilon^{mn} (\tilde{\sigma} \gamma_n \bar{\psi}_k) + \frac{u'}{u''} \epsilon^{mn} (\tilde{\bar{\sigma}} \gamma_n \psi_k) \right) \\ & \left. + \frac{1}{4} \left(\frac{(u')^2}{u''} \psi_m \gamma^m \gamma^n \psi_n + \frac{(\bar{u}')^2}{\bar{u}''} \bar{\psi}_m \gamma^m \gamma^n \bar{\psi}_n \right) \right) \quad (88) \end{aligned}$$

is written in terms of zweibein, gauge-connection and a complex gravitino. The fields ϕ and π are connected with the complex auxiliary field from the superspace approach (cf. [30] for the $N = (1, 1)$ case).

7 Ungauged supergravity

Beside the two versions of minimally gauged $N = (2, 2)$ supergravity discussed so far ungauged versions have been found in the context of superstring compactifications [11–13]. In this section it is shown that models of this type can be obtained in a simple way from the gPSM formulation of the minimally gauged theories.

To formulate an ungauged model we have to get rid of the target space variable π in the Poisson tensors of Sects. 3 and 4. To illustrate this procedure the dilaton prepotential supergravity is taken as an example.

It is straightforward to decouple the additional $U(1)$ charge by choosing the corresponding Casimir function as a new coordinate replacing the coordinate π in (34)-(37). As π appears in the prepotential $u(\phi + i\pi)$ and $\bar{u}(\phi - i\pi)$

the relevant replacement is

$$u(\phi + i\pi) = \hat{u} + \frac{1}{4C} \hat{u}' (\hat{u}\chi^2 - \hat{u}\bar{\chi}^2 + 4iX^a (\chi\gamma_a\gamma_*\bar{\chi})) + \frac{1}{16C} \chi^2 \bar{\chi}^2 (\hat{u}'' + \frac{1}{C} (\hat{u}\hat{u}' - \hat{u}'\hat{u})), \quad (89)$$

Here, $\hat{u}(\phi + iC_\pi)$ and $\hat{u}(\phi - iC_\pi)$ are the prepotentials after replacing π by the Casimir function C_π . For later convenience it is worthwhile to look at the appearance of C inside the new prepotential more in detail. Indeed, one has to insert this prepotential into the expression (38) to compute the remaining conserved quantity. But then C is seen to appear on the right hand side of that equation as well, which could raise the question of the existence of solutions. But this turns out to be a technical problem and does not lead to inconsistencies: C only appears as inverse power in expressions with fermions. By making the split into soul and body $C = C_B + C_S$ a systematic expansion in C_S can be written down and the inverse powers reduce to expressions in $C_B = 8Y - \hat{u}\bar{u}$. The anti-commuting character of the spinors in C_S guarantee that this procedure stops at some point. In this specific case the result is found to be especially simple: A straightforward calculation shows that all new contributions of order $\propto \chi^2 \bar{\chi}^2$ from the expansion in C_S cancel! Thus we just can replace all Casimir functions in (89) by C_B and obtain a closed, but lengthy expression for the remaining Casimir.

Now it is straightforward to reformulate the Poisson tensor with the new variable C_π , which by definition has vanishing components $P^{C_\pi I} = \{C_\pi, X^I\} \equiv 0$. Consequently its gauge potential only appears in the kinetic term $C_\pi d\hat{A}_\pi$. Once C_π is restricted to a constant this becomes an irrelevant total derivative. Thus we may simply redefine $\hat{u}(\phi)$ as a complex prepotential depending on the dilaton alone and drop all reminiscences of C_π . The remaining components of the Poisson tensor still obey the non-linear Jacobi identity (2). The ensuing gPSM action describes an ungauged version of dilaton supergravity. It is important to notice that the basic symmetry principles, local Lorentz invariance encoded in $P^{\phi I}$ and local supersymmetry transformations in $P^{\alpha\beta}$ remain invariant under this change of variables.

Though the explicit calculation of the Poisson tensor is straightforward the lengthy expressions are not very illuminating. Instead the expressions (34)-(37) may be used, if the u and \bar{u} are seen as the abbreviations for (89) and its hermitian conjugate resp. Notice that derivatives with respect to the dilaton have to be taken inside C_B as well.

We add some comments on interesting properties and problems of this model:

1. It is important to realize that the ungauged model is not equivalent to the (twisted-)chiral theories discussed in the previous Sections, although the Poisson tensor of the former *locally* can be obtained from the latter. There are two sources of inequivalence:

- (a) The replacement of the target-space coordinate $\pi \rightarrow C_\pi$ is not defined globally, as can easily be

seen from (69). In particular, the ungauged model only allows for a restricted class of solutions with $C = 0$ but non-vanishing fermion fields. This could have important physical implications as the field configurations with $C = 0$ are candidates for BPS states [32].

- (b) Though $P^{C_\pi I} \equiv 0$ after the replacement $\pi \rightarrow C_\pi$, the models with and without C_π as target-space variable are different: The former still consists of all symplectic leaves labelled by the value of the Casimir function C_π , in the latter case one has to choose a *fixed* value of C_π . Despite the fact that the specific value of C_π is irrelevant it is obvious that the ungauged model now only consists of exactly one symplectic leaf of the full theory⁵. We refer to the discussion of dimensionally reduced Chern-Simons gravity [38–40] as an example of a PSM with $P^{YI} \equiv 0$ in terms of “physical” coordinates for a specific field Y .
2. Only the dilaton prepotential supergravity has been discussed explicitly so far. Obviously, the procedure of decoupling π can be performed for the general model of Sect. 4 as well. Also, the resulting models still follow from a “conformal transformation” of the simplified model with $Z = 0$: Indeed, the model with $Z \neq 0$ can be obtained by “recoupling” of C_π , a subsequent conformal transformation as discussed in Sect. 4 and by decoupling C_π again. The Casimir function C_π , being invariant under the transformation (42) and (43), does not change its value during this procedure and thus we end up in the same symplectic leaf with respect to C_π as we started from. Therefore, it should be possible to circumvent the detour of gauging π again. Notice however, that this more direct transformation cannot be written as a function $Q(\phi)$, solely depending on the dilaton. Rather, the analytic function $Q(\phi + i\pi)$ in (42) and (43) must be rewritten as a function independent of π by substituting $\pi \rightarrow C_\pi$. Then Q will depend on *all* target-space coordinates of ungauged supergravity. Also, the identifications of the new variables according to (42) and (43) will become much more complicated. Alternatively, one could define conformal transformations with a real function $Q(\phi)$ as explained below (44).
3. As in the chiral case we should address mirror symmetry within these models. As a consequence of the absence of additional gaugings one finds that any transformation of the form

$$\begin{aligned} \hat{\chi}^+ &= \cos \alpha^+ \chi^+ + \sin \alpha^+ \bar{\chi}^+, \\ \hat{\chi}^- &= \cos \alpha^- \chi^- + \sin \alpha^- \bar{\chi}^- \end{aligned} \quad (90)$$

leaves invariant the covariant derivatives as well as local supersymmetry transformation. Except for the case $\alpha^\pm = \pi/4$ all transformations (90) are defined globally and have no physical influence. Standard mirror symmetry is defined as the discrete subgroup $\alpha = \pi/2$. But it may well be that for the special case of ungauged

⁵ Of course, both theories still consist of a foliation with respect to C , which has been omitted here for simplicity.

supergravity a generalization of type (90) exists in superspace as well.

4. It remains to check whether this model indeed reproduces the result of [11] once the Lagrange multipliers X^a and the torsion dependent part of ω are eliminated. As can be seen from (89) the ensuing second-order Lagrangian is very complicated. At this point we encounter an additional problem, which is generic for all ungauged supergravity models obtained in this way. Obviously the torsion equation (82) now becomes non-polynomial in X^a due to (89) unless the prepotential is drastically restricted such that all inverse powers of C cancel. Therefore the ungauged version of supergravity presented so far will depend in a non-polynomial way on $\partial_m \phi$ in its second order formulation, though these terms appear in the fermionic potential only. It is emphasized that this behavior does not influence any of the mathematical steps used to eliminate the auxiliary fields. On the contrary, as the elimination procedure does not depend on the details of the Poisson tensor [28,30], the replacement (89) can be made directly in the second order formulation, if X^a is regarded as an abbreviation for (83).

It may be important to point out that a rescaling of the dilatino according to $\chi^\alpha \rightarrow \sqrt{C}\chi^\alpha$ does not remove the inverse powers of C . Indeed, to keep local Lorentz invariance and local supersymmetry in (17), (18) and (20) ψ , X^a and e_a must be rescaled as well. But then inverse powers of C re-emerge in expressions involving u , u' and u'' , esp. in the bosonic potential.

To summarize it was found that ungauged versions of supergravity can be obtained straightforwardly from the (twisted-)chiral versions by decoupling the scalar field π , the partner of the $U(1)$ gauge field B . Nevertheless, symmetry principles from the gPSM formulation are not as restrictive in this case as in $N = (1, 1)$ supergravity and consequently a large number of globally different version of ungauged supergravity were found. Furthermore, lacking a suitable x -space formulation including spinor terms of the action of [11], we do not have any action from superspace at hand that such a result could be compared with.

8 Outlook and conclusions

The present paper shows that $N = (2, 2)$ dilaton supergravity can be formulated in terms of a graded Poisson-Sigma model. The strategy of the construction was motivated by our previous works [29, 30] on $N = (1, 1)$ supergravity: first we solved the model for a very special case, general theories were found by means of conformal transformations, which represent a class of target-space diffeomorphisms in the gPSM. In this way we obtained the explicit Lagrangians for minimally gauged chiral dilaton supergravity. The twisted-chiral version is obtained by mirror symmetry [11], which allows an interpretation as a target-space diffeomorphism as well. Finally it has been outlined how the gPSM result, which represents a

first-order formulation of supergravity with non-vanishing bosonic torsion, can be transformed into a second-order formulation. The latter can be compared with earlier works [1, 2, 10] on (dilaton) supergravity in superspace. The equivalence of the bosonic part of the two dilaton supergravity models, namely the one of [10] and our result, is obvious, but in contrast to [10] the gPSM framework allows a compact, but explicit derivation of *all* spinorial terms as well. In addition the two conserved quantities (Casimir functions) of the theory have been derived explicitly: one can be chosen as the supersymmetrized version of the standard Casimir function of dilaton gravity, coinciding with the ADM mass where such a notion makes sense. The second one represents the charge of the additional $U(1)$ gauge symmetry. As long as the topological character of the theory is not destroyed by the coupling of matter fields, these two quantities essentially describe the complete physical content of the theory.

The result obtained so far motivates numerous applications and generalizations. The similarity to the $N = (1, 1)$ case [28, 30, 32–34] suggests that the advantages of the gPSM framework again enable an exact treatment of the classical theory. This allows the determination of the complete classical solution of the model [37] including all non-trivial fermion contributions. Propagating degrees of freedom may be added by coupling matter fields (cf. [32]). The superspace formulation is certainly simpler to derive invariant Lagrangians, which then can be adjusted to obtain the relevant expression in the gPSM framework. Beside the classical considerations, it must be possible to quantize pure $N = (2, 2)$ dilaton supergravity too in a non-perturbatively exact way when formulated as a gPSM. Matter interactions still can be treated perturbatively (cf. [33,34] for the $N = (1, 1)$ case). As for $N = (1, 1)$ a classification of all BPS states should be possible.

A yet different aspect is the deformation of $N = (2, 2)$ dilaton supergravity to models exhibiting only $N = (1, 1)$ invariance. In the limit where the $N = (2, 2)$ invariance is recovered solitonic states may appear. Though it is not straightforward to realize kink solutions with the dilaton alone [38, 39] extensions within supergravity are possible [40]. However, the situation could change with $N = (2, 2)$ supergravity as the field content encompasses an additional scalar field.

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Appendix A: Notations and conventions

The conventions are identical to [28, 35], where additional explanations can be found.

Indices chosen from the Latin alphabet are generic (upper case) or (lower case) refer to commuting objects, Greek indices are anti-commuting ones. Holonomic coordinates

are labeled by M, N, O etc., anholonomic ones by A, B, C etc., whereas I, J, K etc. are general indices of the gPSM:

$$X^I = (X^\phi, X^\pi, X^a, X^\alpha, X^{\bar{\alpha}}) = (\phi, \pi, X^a, \chi^\alpha, \bar{\chi}^\alpha) \quad (\text{A.1})$$

$$A_I = (A_\phi, A_\pi, A_a, A_\alpha, A_{\bar{\alpha}}) = (\omega, B, e_a, \psi_\alpha, \bar{\psi}_\alpha) \quad (\text{A.2})$$

The summation convention is always $NW \rightarrow SE$, e.g. for a fermion χ : $\chi^2 = \chi^\alpha \chi_\alpha$. Our conventions are arranged in such a way that almost every bosonic expression is transformed trivially to the graded case when using this summation convention and replacing commuting indices by general ones. This is possible together with exterior derivatives acting *from the right*, only. Thus the graded Leibniz rule is given by

$$d(AB) = AdB + (-1)^B (dA)B. \quad (\text{A.3})$$

In terms of anholonomic indices the metric and the symplectic 2×2 tensor are defined as

$$\eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A.4})$$

$$\epsilon_{ab} = -\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (\text{A.5})$$

The metric in terms of holonomic indices is obtained by $g_{mn} = e_n^a e_m^b \eta_{ab}$ and for the determinant the standard expression $e = \det e_m^a = \sqrt{-\det g_{mn}}$ is used. The volume form reads $\epsilon = \frac{1}{2} \epsilon^{ab} e_b \wedge e_a$; by definition $*\epsilon = 1$.

The γ -matrices are used in a chiral representation:

$$\gamma^0_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^1_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (\text{A.6})$$

$$\gamma_{*\alpha}{}^\beta = (\gamma^1 \gamma^0)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A.7})$$

Covariant derivatives of anholonomic indices with respect to the geometric variables $e_a = dx^m e_{am}$ and $\psi_\alpha = dx^m \psi_{\alpha m}$ include the two-dimensional spin-connection one form $\omega^{ab} = \omega \epsilon^{ab}$. When acting on lower indices the explicit expressions read ($\frac{1}{2} \gamma_*$ is the generator of Lorentz transformations in spinor space):

$$(De)_a = de_a + \omega \epsilon_a{}^b e_b \quad (D\psi)_\alpha = d\psi_\alpha - \frac{1}{2} \omega \gamma_{*\alpha}{}^\beta \psi_\beta \quad (\text{A.8})$$

Dirac conjugation is defined as $\bar{\chi}^\alpha = \chi^\dagger \gamma_0$. Written in components of the chiral representation

$$\chi^\alpha = (\chi^+, \chi^-), \quad \chi_\alpha = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \quad (\text{A.9})$$

the relation between upper and lower indices becomes $\chi^+ = \chi_-$, $\chi^- = -\chi_+$. Dirac conjugation follows as

$\bar{\chi}_- = \chi_-^*$, $\bar{\chi}_+ = -\chi_+^*$, i.e. for Majorana spinors χ_- is real while χ_+ is imaginary.

For two gauge-covariant Dirac spinors χ_α and λ_α the combinations

$$\chi\lambda, \quad \chi\gamma_*\lambda, \quad \bar{\chi}\gamma^a\lambda \quad (\text{A.10})$$

and their hermitian conjugates are gauge invariant for chiral gaugings, while

$$\bar{\chi}\lambda, \quad \bar{\chi}\gamma_*\lambda, \quad \bar{\chi}\gamma^a\lambda \quad (\text{A.11})$$

are invariant for twisted-chiral gaugings. Note that in the latter case the gravitino ψ_α transforms under gauge transformations as $\bar{\chi}_\alpha$. Thus in (A.11) the bilinear invariants of a gravitino and a dilatino are obtained by substituting $\lambda \rightarrow \psi$.

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